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FITTING A POLYNOMIAL TO DATA IN THE PRESENCE OF NOISE.(U)

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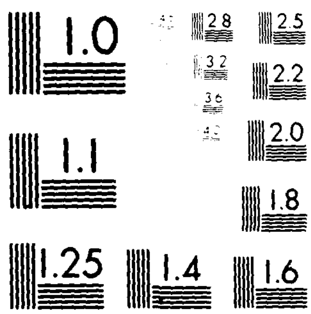

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Fitting a Polynomial to Data in the Presence of Noise

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PREFACE

This work was sponsored by NAVSEA 63R, Daniel Porter, Program Manager, and was conducted within NUSC Code 321 under J.O. A65400; Norman Owsley was the Principal Investigator for the NUSC project in Multifunction Adaptive Processing.

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REVIEWED AND APPROVED: 6 February 1982

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4. FITTING A POLYNOMIAL TO DATA IN THE PRESENCE OF NOISE

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19. Signal processing
Polynomial models

20. Polynomial models arise in numerous applications in signal processing dealing with the estimation of continuous functions of either space or time. The requirement to interpolate between discrete noisy data points is of particular interest. In this report, the general-order polynomial time-varying coefficient-estimation problem is formulated and a structure is indicated for implementation of the coefficient-estimation process. Both the maximum likelihood (ML) and maximum a posteriori (MAP) estimation coefficient estimators are derived and the corresponding analyses of (over)

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variance are obtained. The cases of measurement data consisting of samples of either the polynomial directly or the polynomial slope in the presence of additive noise are considered. Finally, expressions for the sensitivity to uncompensated measurement bias of the estimator standard deviations are obtained.

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1.0 INTRODUCTION

Polynomials provide commonly used models for functions representing processes of interest in space- and time-series analysis. Specifically, the polynomial provides a convenient method for data interpolation and smoothing. Of particular interest here is the N-th order polynomial, which is linear in its coefficients. The general problem allows for time-varying coefficients for which prior statistical knowledge of the polynomial coefficients as random processes is available either through physical modeling or actual measurements.

For the time-invariant polynomial, two types of estimators are considered. First, the maximum likelihood (ML) estimator, which assumes the coefficients to be nonrandom (deterministic) but unknown, is considered. Then, the maximum a posteriori (MAP) estimator, which assumes the coefficients to be random variables, is considered. The MAP estimator requires prior knowledge of the mean values and covariance matrix for the random polynomial coefficients. Both the ML and MAP estimators assume the polynomial coefficients to be constant over the temporal interval for which measurements are available. For time-variable coefficients, a recursive (Kalman) filter realization of the MAP coefficient estimator is presented.

In practice, it is frequently the case that the data to which the polynomial model is to be fitted consists either of samples of the underlying process in a Cartesian (x,y) coordinate system plus a measurement noise or samples of the slope (derivative) of the underlying process plus a corresponding disturbance. For both types of fit data, the measurement noise is assumed to be zero mean, uncorrelated, and Gaussian with known variance.

Examples of both Cartesian (amplitude) and slope-sampled data occur in position tracking and phase tracking. In the position-estimation problem, amplitude sampling would correspond to having fit data in Cartesian coordinate form, i.e., noisy measurements of the y-coordinate as a function of the x-coordinate. Slope sampling would correspond to having noise-corrupted angle measurements of the tangent to the underlying function relative to a coordinate system that allows a small-angle linearization assumption. The small-angle assumption allows the function derivative and function tangent angle to be equated. A specific example here might be the reconstruction of a continuous spatial track approximated by a polynomial, given data consisting of either compass-heading measurements (slope) or latitude/longitude measurements (amplitude) at discrete points along the track. With respect to the phase-tracking example, amplitude sampling would correspond to having discrete time samples of the noise-corrupted phase process and slope sampling would correspond to having samples of the time derivative of the phase process, i.e., instantaneous frequency, from which to reconstruct the continuous phase function.

After introducing the basic polynomial-fit problem in noise, the next section develops the ML estimator. The ML estimator variance is obtained and

the question of biased measurements is examined. The following section presents a similar development of the MAP estimator. Next, some numerical results are presented and some useful expressions are obtained for the second order polynomial fit. Finally, a recursive estimation scheme for tracking time-varying polynomial coefficients is formulated.

2.0 THE (POLYNOMIAL FIT) PROBLEM

Let the N-th order polynomial, $y(s)$, be represented on the interval $0 \le s \le L$ by

$$y(s) = \sum_{n=1}^{N} a(n)s^n . \tag{1}$$

Let θ be an N-vector of polynomial coefficients with a matrix transpose, θ^T, defined by

$$\theta^T = [a(1)\ a(2)\ \cdot\ \cdot\ \cdot\ a(N)] \tag{2}$$

Similarly, let a vector, $\Phi_1(s)$, be defined by

$$\Phi_1^T(s) = \left[s\ s^2\ \cdot\ \cdot\ \cdot\ s^N\right],\quad 0 \le s \le L \tag{3}$$

If we use equations (2) and (3), $y(s)$ becomes

$$y(s) = \theta^T\Phi_1(s) \tag{4}$$

The derivative of $y(s)$ is given by

$$\begin{aligned} d(s) &= dy(s)/ds \\ &= \sum_{n=1}^{N} n a(n) s^{n-1} \\ &= \theta^T\Phi_2(s) , \end{aligned} \tag{5}$$

where the vector, $\Phi_2(s)$ is defined by its transpose as

$$\Phi_2^T(s) = \left[1\ 2s\ \cdot\ \cdot\ \cdot\ Ns^{N-1}\right] \tag{6}$$

The measurement data

$$z_1(\theta,k) = y(s_k) + v_1(\theta,k) \quad \text{(amplitude)} \tag{7}$$

and

$$z_2(m,k) = d(x_m) + v_2(m,k) \quad \text{(slope)} \tag{8}$$

for $m = 1,2,\ldots,M$ position samples and $k = 1,2,\ldots,K$ time samples are available when the N-th order polynomial is to be fit to amplitude- and slope-sampled data, respectively. The M sample points, x_m, $m = 1,2,\ldots,M$, are assumed to be known. The index k is a discrete time-sample number index for which a measurement interval consisting of K samples is available.

In matrix notation, the measurement vector, $\mathbf{z}_i(k)$, is defined by

$$\mathbf{z}_i^T(k) = [z_i(1,k)\; z_i(2,k)\; \cdots\; z_i(M,k)] \tag{9}$$

at time k with $i = 1$ and $i = 2$ for amplitude and slope data fitting, respectively.

The measurement vector, $\mathbf{z}_i(k)$, at time k is defined further by

$$\mathbf{z}_i(k) = \mathbf{H}_i \boldsymbol{\theta} + \mathbf{v}_i(k) , \tag{10}$$

where $\mathbf{H}_1$ is an M-by-N matrix given by

$$\mathbf{H}_1 = \begin{bmatrix} x_1 & x_1^2 & \cdots & x_1^N \\ x_2 & x_2^2 & \cdots & x_2^N \\ & \vdots & & \\ x_M & x_M^2 & \cdots & x_M^N \end{bmatrix} \tag{11}$$

when the polynomial is fit to Cartesian data, and

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 2x_1 & \cdots & Nx_1^{N-1} \\ 1 & 2x_2 & \cdots & Nx_2^{N-1} \\ & \vdots & & \\ 1 & 2x_M & \cdots & Nx_M^{N-1} \end{bmatrix} \tag{12}$$

when a slope fit is required. The measurement noise vector, $\mathbf{v}_i(k)$, is a Gaussian vector sequence with mean $\mathbf{b}_i$ and covariance (matrix) given by the expectation

$$E\left\{[\mathbf{v}_i(p) - \mathbf{b}_i][\mathbf{v}_i(q) - \mathbf{b}_i]^T\right\} = \sigma_i^2 \, \mathbf{I}_M \, \delta(p - q) \tag{13}$$

In equation (13), σ_i^2 is the measurement noise variance, $\mathbf{I}_M$ is an M-by-M identity matrix, and

$$\delta(p - q) = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases} \quad (14)$$

With [illegible] definitions, the problem of fitting a polynomial to data in the presence of noise can be stated as:

1. Given a sequence of K noise-corrupted measurement vectors, $\underline{z}_i(k)$, k = 1, 2, ..., K (equation 10), for i = 1 (polynomial) and i = 2 (polynomial slope),

2. Find an estimate of the vector of polynomial coefficients, $\underline{a}$, and the corresponding polynomial shape,

$$y(k) = \underline{a}^T \underline{p}_1(k) .$$

As a particular estimator performance metric, the ability to estimate the deviation of the polynomial from a straight line is of special interest. The scalar metric selected for this purpose is the deflection of the polynomial at its midpoint, k = K/2, from a straight line passing through the end points at k = 0, K. The estimator for the midpoint deflection, referred to here as distortion, is

$$\epsilon = [y(k) - [y(K)/K]k]_{(k = K/2)}$$

$$= \underline{a}^T \left[\underline{p}_1\left(\frac{K}{2}\right) - \frac{1}{2}\underline{p}_1(K)\right] \quad (15)$$

[illegible]

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$$[illegible] \quad (16)$$

[illegible]

with respect to variations in θ. If this maximization is performed, the corresponding maximum extreme value for θ is the estimator

$$\theta_{ML} = \left[H^T H\right]^{-1} H^T \left[\frac{1}{K} \sum_{k=1}^{K} z(k) - b\right] . \tag{17}$$

Because $E\{z(k)\} = H\theta + b$, it is straightforward to show that the expected value of θ_{ML} is θ, which establishes θ_{ML} as an unbiased estimator. The estimator covariance matrix is given by

$$P_{ML} = E\left\{(\theta_{ML} - \theta)(\theta_{ML} - \theta)^T\right\}$$

$$P_{ML} = \frac{\sigma^2}{K}\left[H^T H\right]^{-1} \tag{18}$$

Because θ_{ML} provides an unbiased estimate, the lower bound for the variance of any unbiased estimate of the n-th polynomial coefficient is established by the n-th diagonal element of the inverse of the Fisher information matrix,[3] $\mathcal{J}_{ML}$, as

$$E\left\{\left[\hat{a}_{ML}(n) - a(n)\right]^2\right\} \geq \left[\mathcal{J}_{ML}\right]^{-1}_{nn} \tag{19}$$

The pq-th element in the $\mathcal{J}$ matrix is

$$\left[\mathcal{J}_{ML}\right]_{pq} = -E\left\{ \frac{[illegible]}{\partial a(p)\, \partial a(q)} \right\}$$

$$= \frac{K}{\sigma^2}\left[H^T H\right]_{pq} \tag{20}$$

Thus, it follows that

$$\left[\mathcal{J}_{ML}\right]^{-1} = \frac{\sigma^2}{K}\left[H^T H\right]^{-1} , \tag{21}$$

which is precisely the error covariance matrix for the ML estimator established in equation (18). Therefore, the ML estimator, θ_{ML}, of equation (17) reaches the minimum variance bound.

As a final issue, the question of measurement bias, which is unaccounted for in the ML estimation procedure, is considered. Suppose that the measurement noise vector, $V(k)$, is assumed to be zero mean when, in fact, it has mean b. The estimator that results in this situation is

$$\theta_{ML_2} = \left[H^T H\right]^{-1} H^T \left[\frac{1}{K} \sum_{k=1}^{K} z(k)\right] . \tag{22}$$

with mean value

$$E\left[\hat{\theta}_{ML_E}\right] = \theta + \left[H^T H\right]^{-1} H^T \mathbf{b} \tag{23}$$

and correlation matrix

$$R_{ML_E} = R_{ML} + \left[H^T H\right]^{-1} H^T \mathbf{b}\mathbf{b}^T H \left[H^T H\right]^{-1} . \tag{24}$$

Thus, in accordance with equation (24), an increased mean-squared error (MSE) results in the biased estimator case. This expression is used in a following section to evaluate the standard deviation for the distortion estimate, which is a linear combination of the estimated polynomial coefficients.

4.0 MAXIMUM A POSTERIORI (MAP) ESTIMATION

The MAP estimator, $\hat{\theta}_{MAP}$, of a random polynomial coefficient vector is obtained by maximizing the logarithm of the posteriori density function

$$\ell[\theta / \mathbf{g}(k),\ k = 1, 2, \ldots, K] + C_1 = \log p[\mathbf{g}(k),\ k = 1, 2, \ldots, K/\theta]$$

$$+ \log p(\theta) \tag{25}$$

For θ a Gaussian random vector with zero mean and covariance matrix R_θ, equation (25) becomes

$$\ell + C_2 = -\frac{1}{2\sigma^2} \sum_{k=1}^{K} \left\| \mathbf{g}(k) - H\theta - \mathbf{b} \right\|^2 - \frac{1}{2} \theta^T R_\theta^{-1} \theta \tag{26}$$

This, in turn, is maximized with respect to θ by setting θ equal $\hat{\theta}_{MAP}$, where

$$\hat{\theta}_{MAP} = \left(H^T H + \frac{\sigma^2}{K} R_\theta^{-1}\right)^{-1} H^T \left[\frac{1}{K} \sum_{k=1}^{K} \mathbf{g}(k) - \mathbf{b}\right] \tag{27}$$

As with the ML estimator, $\hat{\theta}_{MAP}$ is unbiased because $E\{\hat{\theta}\} = \theta$. It is straightforward, but tedious, to show that the covariance matrix for $\hat{\theta}_{MAP}$ is R_{MAP}, given by

$$R_{MAP} = E\left[(\hat{\theta}_{MAP} - \theta)(\hat{\theta}_{MAP} - \theta)^T\right]$$

$$= \left[H^T H + \frac{\sigma^2}{K} R_\theta^{-1}\right]^{-1} H^T \left[H R_\theta H^T + \frac{\sigma^2}{K} I_M\right] H \left[H^T H + \frac{\sigma^2}{K} R_\theta^{-1}\right]^{-1} \tag{28a}$$

$$- \left[H^T H + \frac{\sigma^2}{K} R_\theta^{-1}\right]^{-1} H^T H R_\theta - R_\theta H^T H \left[H^T H + \frac{\sigma^2}{K} R_\theta^{-1}\right]^{-1} + R_\theta .$$

which has the alternate form

$$\mathbf{R}_{MAP} = \mathbf{R}_A - \mathbf{R}_A \mathbf{H}^T\left(\mathbf{H}\mathbf{R}_A\mathbf{H}^T + \frac{\sigma^2}{K}\mathbf{I}_M\right)^{-1}\mathbf{H}\mathbf{R}_A , \tag{28b}$$

where $\mathbf{I}_M$ is the M-by-M identity matrix. Thus, as the number of measurements increases to the limit, there results from equation (28a)

$$\lim_{K\to\infty} \mathbf{R}_{MAP} = \mathbf{0} . \tag{29}$$

In the random-parameter case, the total information matrix is obtained by appending the inverse matrix $\mathbf{R}_A^{-1}$, which represents prior knowledge, to the information matrix for the nonrandom ML parameter estimator. Accordingly,[1]

$$\begin{aligned} \mathbf{J}_{MAP} &= \mathbf{J}_{ML} + \mathbf{J}_{PRIOR} \\ &= \frac{K}{\sigma^2}\mathbf{H}\mathbf{H}^T + \mathbf{R}_A^{-1} , \end{aligned} \tag{30}$$

which provides the covariance matrix bound

$$\begin{aligned} \mathbf{J}_{MAP}^{-1} &= \mathbf{R}_A - \mathbf{R}_A \mathbf{H}^T\left(\mathbf{H}\mathbf{R}_A\mathbf{H}^T + \frac{\sigma^2}{K}\mathbf{I}_M\right)^{-1}\mathbf{H}\mathbf{R}_A \\ &= \mathbf{R}_{MAP} . \end{aligned} \tag{31}$$

Therefore, the lower bound on the variance of an unbiased MSE estimator is satisfied by the MAP estimator of the n-th polynomial coefficient, a(n). This bound is given by

$$E\left\{a_{MAP}^2(n)\right\} \geq \left(J_{MAP}^{-1}\right)_{nn} , \tag{32}$$

which is satisfied by the equality.

5.0 THE SECOND-ORDER POLYNOMIAL

In this section, the important second-order polynomial model

$$y(x) = a(1)x + a(2)x^2 , \quad (0 \leq x \leq L) \tag{33}$$

is examined in detail with respect to the estimation of the coefficients a(1) and a(2). The estimator error variance and corresponding lower variance bound are examined with respect to M, the number of spatial-sample points $x_1, x_2, \ldots, x_M$ available and K, the number of time-sample points.

The M positional sample points are assumed to be uniformly spaced over the interval (0,L) at

$$x_m = \frac{m - 1}{M - 1} L , \quad m = 1,2, \cdots ,M . \tag{34}$$

For the ML estimation case, the midpoint-distortion estimator has variance, from equations (15) and (18),

$$\sigma^2_{y_{ML}}\left(x = \frac{L}{2}\right)_1 = \left[\mathbf{P}\left(\frac{L}{2}\right) - \frac{1}{2}\mathbf{P}(L)\right]^T \mathbf{R}_{ML_1}\left[\mathbf{P}\left(\frac{L}{2}\right) - \frac{1}{2}\mathbf{P}(L)\right] \tag{35}$$

$$\boxed{\sigma^2_{y_{ML}}\left(x = \frac{L}{2}\right)_1 = \frac{15}{2}\,\frac{\sigma_1^2}{K}\,\frac{(M - 1)^3(2M - 1)}{M(M + 1)(M - 2)(3M^2 - 3M + 2)}} \tag{36}$$

For amplitude measurement data. In a similar manner,

$$\boxed{\sigma^2_{y_{ML}}\left(x = \frac{L}{2}\right)_2 = \frac{3}{16}\,\frac{\sigma_2^2}{KM}\,L^2\,\frac{M - 1}{M + 1}} \tag{37}$$

For slope measurement data. The midpoint-distortion estimator is a linear combination of the ML polynomial coefficient estimators, which individually have been shown to reach the so-called Cramer Rao (CR) lower variance bound. Thus, the distortion ML estimator also performs at the CR lower bound on variance.

The MAP estimator, $\mathbf{a}_{MAP}$, allows the covariance matrix, $\mathbf{R}_{MAP}$, to be expressed explicitly in terms of the measurement variance σ^2 and the parameters W and K. In particular, the MAP midpoint-distortion estimator, using slope measurements, is given by

$$\sigma^2_y(L/2)_2 = \frac{rpL^4}{16}\left[1 - \frac{kpr}{\sigma_2^2}\left\{\frac{2}{3}ML^2\,\frac{2M - 1}{M - 1}\right.\right.$$

$$\left.\left. - \frac{\frac{2}{3}L^2 \cdot \frac{2M - 1}{M - 1}\left[1 - \left(M + \frac{\sigma_2^2}{pK}\right)\left(\frac{2}{3} \cdot \frac{2M - 1}{M - 1}\right)\right] - \left(\frac{\sigma_2^2}{prK}\right)}{1 - \left(\frac{M + \sigma_2^2}{pK}\right)\left[\frac{2}{3} \cdot \frac{2M - 1}{M - 1} + \frac{\sigma_2^2}{prK(LM^2)}\right]}\right\}\right] \tag{38}$$

where the coefficient covariance matrix $\mathbf{R}_A$ is defined as

$$\mathbf{R}_A = p\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} . \tag{39}$$

The corresponding variance using amplitude measurements is more complicated and, therefore, provides little additional insight into critical parameter dependence. It is, therefore, not included here.

The sensitivity of the ML estimator to measurement bias error for the second-order system is of particular interest. From equation (24), the additional MSE contribution for the midpoint distortion due to an uncompensated measurement bias vector, $\mathbf{b}$, is

$$\sigma^2_{y_{ML}}\left(\frac{L}{2}\right) = \left|\left[\mathbf{p}\left(\frac{L}{2}\right) - \frac{1}{2}\ \mathbf{p}(L)\right]^T \left(\mathbf{H}^T\mathbf{H}\right)^{-1} \mathbf{H}^T\mathbf{b}\right|^2 . \tag{40}$$

The first type of bias considered is termed "single-point bias," wherein

$$\mathbf{b}^T = [0 \cdot \cdot \cdot 0\ b\ 0 \cdot \cdot \cdot 0] , \tag{41}$$

such that only the m-th element in the bias vector is nonzero with bias b. For the slope-measurement case, the resulting bias variance in the distortion estimator is

$$\sigma^2_{y_{ML}}\left(\frac{L}{2}\right)_b = \left[\frac{3bL(M - 2m + 1)}{4M(M + 1)}\right]^2 . \tag{42}$$

The corresponding sensitivity of the standard deviation to the bias is, therefore,

$$\frac{\partial \sigma_{y_{ML}}\left(\frac{L}{2}\right)_b}{\partial b} = \left[\frac{135L}{\pi M(M + 1)} \left|M - 2m + 1\right|\right] \ \frac{\text{units of length}}{\text{degree}} . \tag{43}$$

An examination of equation (43) indicates that the least degrading region for single-point bias errors to occur is near the center of the measurement aperture. In fact, for the point $m = (M + 1)/2$, the error is precisely zero. On the other hand, single-point bias errors near the measurement aperture extremities are most degrading. Clearly, having a large number of sensors serves to reduce the single-point bias errors by averaging.

The second type of bias error is referred to as "uniform bias," wherein

$$\mathbf{b}^T = b[1\ 1 \cdot \cdot \cdot 1] , \tag{44}$$

that is, all heading-angle measurements are equally biased by an amount b. In this instance, it is straightforward to show that the bias MSE from equation (40) is

$$\sigma^2_{y_{ML}}\left(\frac{L}{2}\right) = 0 \tag{45}$$

for the slope-measurement process. This result shows that if equal bias exists in the slope measurement at all M measurement points, the net effect in terms of estimating the quadratic component of polynomial shape distortion is zero. Obviously, an error would occur in estimating the linear (rotational) component of the polynomial shape. This rotational error is linearly related to the bias angle b. Finally, if all M of the slope-measurement sensors have a random error, b_m; $m = 1,2,\ldots,M$, which are zero mean, uncorrelated, and identically distributed with variance σ^2_b, then, by superposition, equation (42) yields

$$\sigma^2_{y_{ML}}\left(\frac{L}{2}\right)_b = \frac{3}{16}\,\frac{\sigma^2_b}{M}\,L^2\,\frac{M - 1}{M + 1} . \tag{46}$$

Notice the similarity between the unbiased ML estimator variance of equation (37) and the result in equation (46). However, the bias error is not reducible by time averaging because it is a static bias.

6.0 NUMERICAL RESULTS

Simple variance expressions for the polynomial midpoint-distortion estimators are difficult to obtain for polynomials of order $N \geq 3$. Accordingly, in this section, some numerical results are presented that illustrate the dependence of the midpoint-distortion estimator standard deviation on the number of equally spaced measurement (sensor) points M, which are uniformly distributed at points $x_m = (m - 1)L/(M - 1)$, $m = 1,2,\ldots,M$, and the order of the polynomial N. For the MAP estimator, the prior knowledge, which consists of the polynomial-coefficient covariance matrix, is given by

$$\mathbf{R}_A = \begin{bmatrix} 3.637 \times 10^{-5} & 0 & 0 \\ 0 & 4.4 \times 10^{13} & 0 \\ 0 & 0 & 2.0 \times 10^{-18} \end{bmatrix} . \qquad (47)$$

The estimator standard deviation versus number of measurement points, M, for both Cartesian (amplitude-) and slope-measurement data fitting are presented in figures 1 and 2, respectively. A single time sample, K, is assumed. The measurement noise variances assumed are

$$\sigma_1^2 = 1.0 \text{ m}^2$$

for Cartesian measurements and

$$\sigma_2^2 = 0.228 \times 10^{-5} \text{ (radians)}^2$$

for slope measurements.

A general observation from these results is that the order of the fit polynomial should be as low as possible while remaining consistent with the underlying physical process being modeled. The ML process degrades less rapidly with increasing polynomial order than the MAP process, at least for the covariance matrix given in equation (47). These qualitative statements are quantified by the results presented in figure 3 for Cartesian (amplitude) measurement data and figure 4 for slope measurements. For both estimators, as the number of measurement points, M, increases, the variance decreases. Both estimates, in fact, are asymptotically equivalent as either the number of measurement points, M, or the number of time samples, K, becomes infinite.

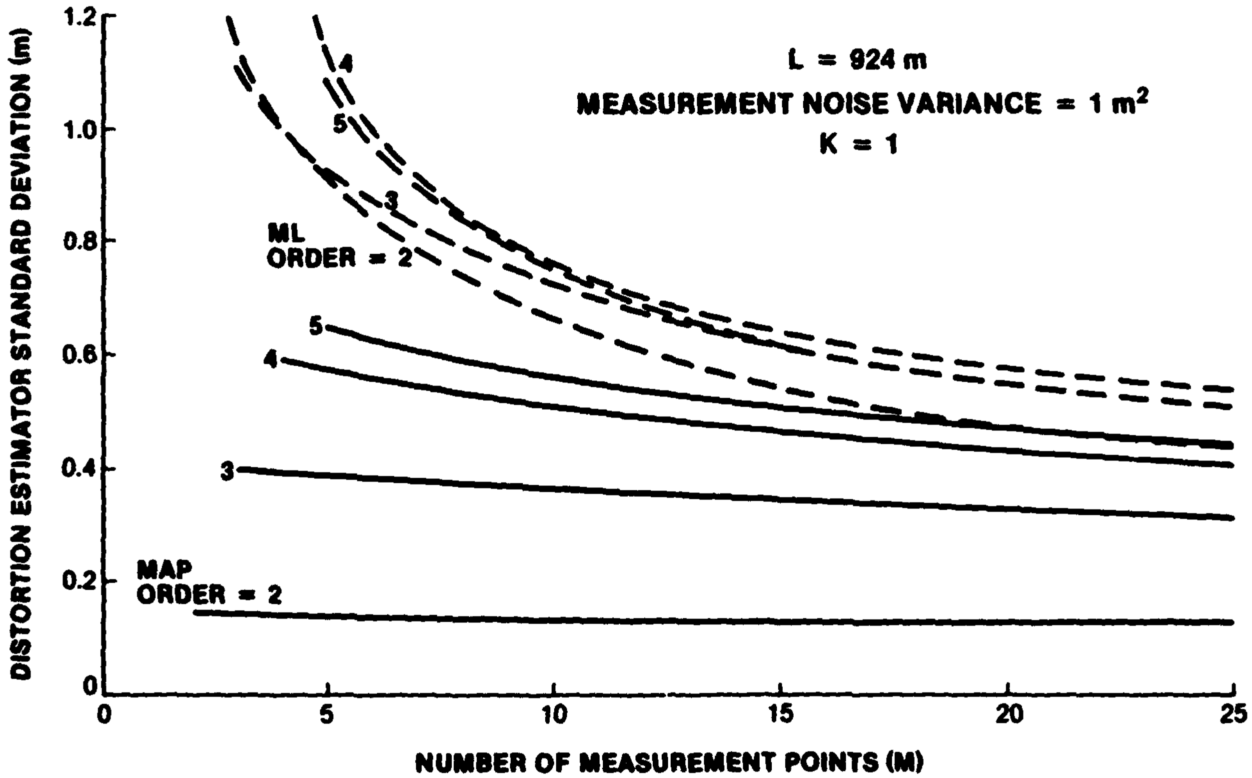

Figure 1. Distortion Estimator Standard Deviation Versus Number of Measurement Points (M), Amplitude Fit

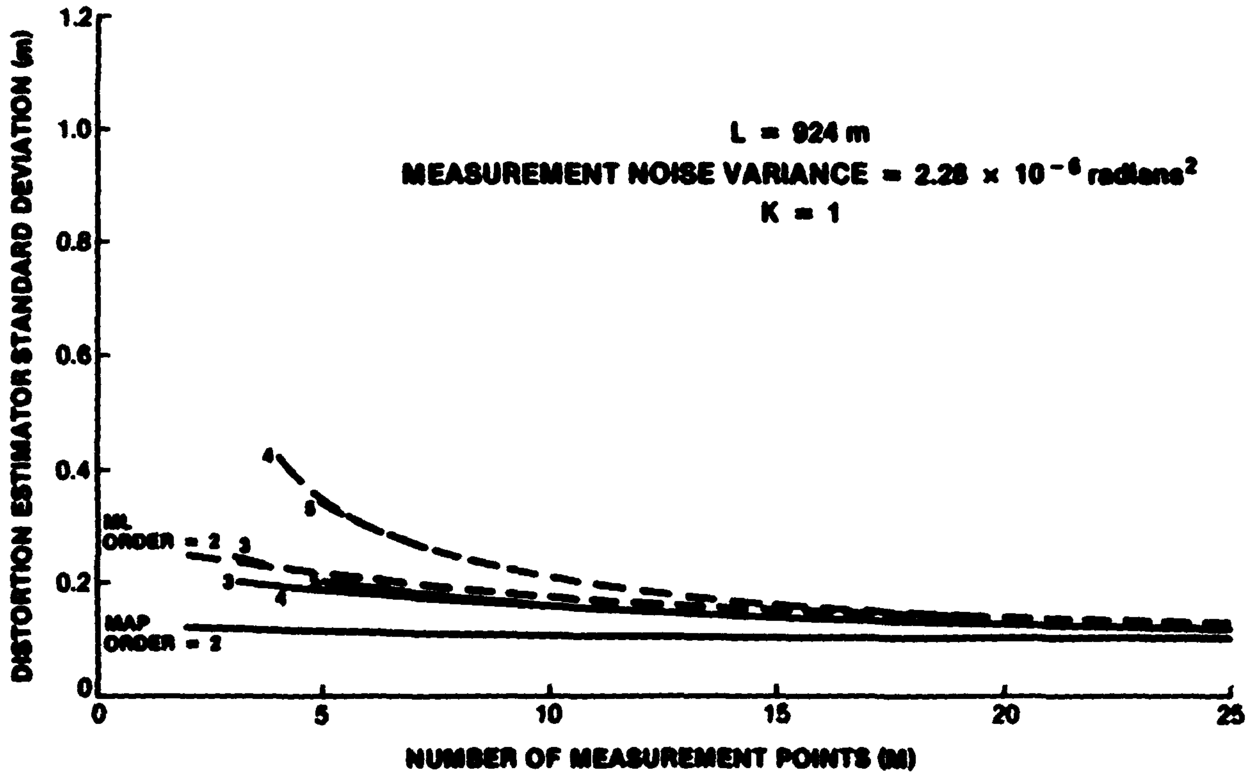

Figure 2. Distortion Estimator Standard Deviation Versus Number of Measurement Points (M), Slope Fit

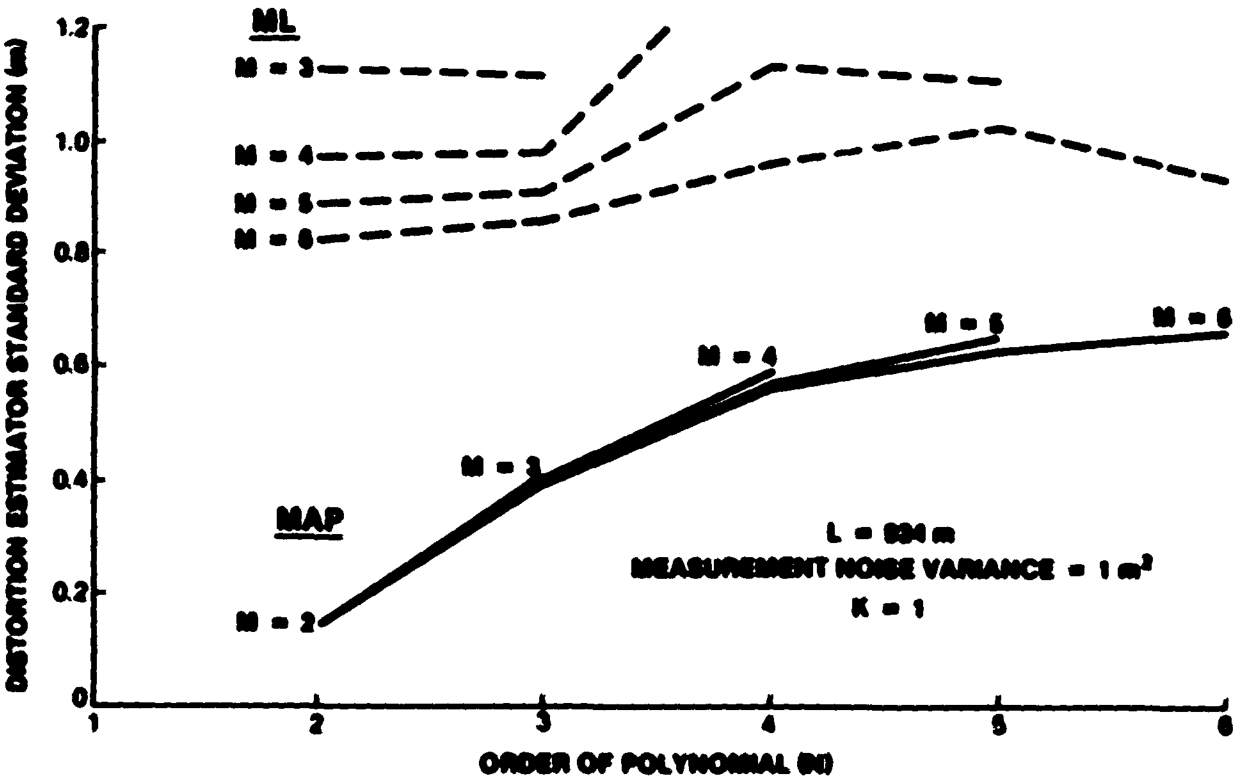

Figure 3. Distortion Estimator Standard Deviation Versus Polynomial Order (N) for Fixed M, Amplitude Fit

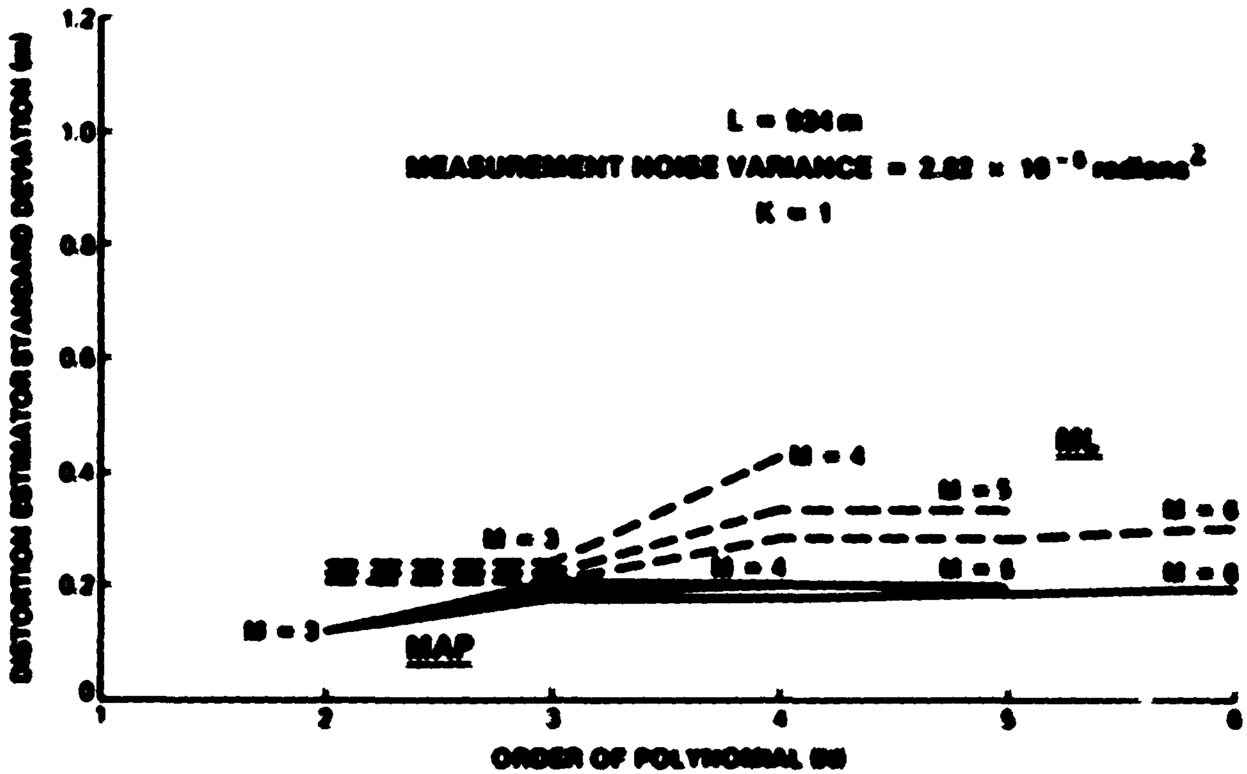

Figure 4. Distortion Estimator Standard Deviation Versus Polynomial Order (N) for Fixed M, Slope Fit

7.0 TIME-VARYING POLYNOMIAL COEFFICIENTS

In the N-th order polynomial of equation (1), the polynomial coefficients $a(n)$, $n = 1, 2, \ldots, N$, are assumed to be constant over the measurement interval consisting of K independent samples. In this section, it is desired to allow these coefficients to be time varying and to devise an appropriate continuously operating recursive estimation scheme to replace the ML and MAP "block" estimation schemes, which have been developed for the constant coefficient case. Accordingly, let the time-varying N-vector $\mathbf{a}(t)$ be a continuous zero-mean vector random process where

$$\mathbf{a}^T(t) = [a(1,t)\ a(2,t) \cdots a(N,t)] \ , \tag{48}$$

such that $a(n,t)$ is the coefficient in the n-th term of the time-varying polynomial

$$y(x,t) = \mathbf{a}^T(t)\mathbf{x}(x)$$

$$= \sum_{n=1}^{N} a(n,t)x^n \ . \tag{49}$$

The first-order stochastic differential equation

$$\frac{d\mathbf{a}(t)}{dt} = -\mathbf{F}\mathbf{a}(t) + \mathbf{w}(t) \tag{50}$$

is introduced, where $d\mathbf{a}(t)/dt$ is an N-vector with n-th element $da(n,t)/dt$, $\mathbf{F}$ is the N-by-N system (coupling) matrix and $\mathbf{W}(t)$ is a stationary, zero-mean, and uncorrelated Gaussian vector random process at time t with covariance matrix

$$E\left\{\mathbf{w}(t)\mathbf{w}(t+\tau)^T\right\} = \mathbf{Q}\delta(\tau) \ . \tag{51}$$

The quantity $\delta(\tau)$ is the unit impulse function at $\tau = 0$ and the covariance matrix $\mathbf{Q}$ is assumed to be known. The system matrix, $\mathbf{F}$, is assumed either to be known from dynamic modeling, empirically determined, or adaptively estimated.[2]

In discrete time-sampled data form, if the product between the time-sample interval T and any term f_{ij} in the system matrix $\mathbf{F}$ is much less than one, there results

$$\mathbf{a}(k+1) = \mathbf{\Phi}(k+1,k)\mathbf{a}(k) + T\mathbf{w}(k) \ . \tag{52}$$

In equation (52),

$$\mathbf{a}(k) = \mathbf{a}(kT) \ ,$$

$$\mathbf{w}(k) = \mathbf{w}(kT) \ ,$$

and

$$\Phi(k+1,k) = \left[I_N + FT\right] \tag{53}$$

is the state-transition matrix, where I_N is an N-by-N identity matrix. As in the constant coefficient case, the measurement vector

$$z_i(k) = H_i a(k) + v_i(k) \tag{54}$$

at time $t = kT$ is available. The only difference between the measurement vector of equation (54) and that in equation (10) is that the coefficient vector is time varying in equation (54), i.e., $a(k)$ is a function of the discrete time index, k. Again, the index i stipulates that either Cartesian data ($i = 1$) or slope data ($i = 2$) is available as measurement data.

Equations (52), (53), and (54), in conjunction with well-known procedures for linear Kalman filter design, specify the recursive sequence presented in table 1. Because this filtering operation is thoroughly developed elsewhere,[7] only the implementation algorithms are presented here for completeness. The exception to the lack of detail in the Kalman-filter development concerns the analysis of steady-state estimator variance. In particular, if $\hat{a}(t)$ is the coefficient vector estimator at time t, then

$$P(t) = E\left\{[\hat{a}(t) - a(t)][\hat{a}(t) - a(t)]^T\right\} \tag{55}$$

is the corresponding error covariance matrix at time t.

The covariance matrix trajectory is solved by the matrix Riccati equation with $P(0) = P_0$,

$$\dot{P} = FP + PF^T + Q - PH^TR^{-1}HP , \tag{56}$$

where the measurement-type index, i, has been deleted for e[illegible]

Of special interest is the steady-state ($\dot{P} = 0$) perf[illegible] the recursive filter. For this case, the matrix algebraic Ricat[illegible] can be solved for the steady-state polynomial coefficient covar[illegible], P, by completing the square. Writing equation (56) as

$$PH^TR^{-1}HP - FP - PF^T = Q , \quad (\dot{P} = 0) , \tag{57}$$

which, in turn, can be rewritten as

$$\begin{aligned} &\left[P - F(H^TR^{-1}H)^{-1}\right]H^TR^{-1}H\left[P \right. \\ &\left. - F(H^TR^{-1}H)^{-1}\right]^T = Q + F(H^TR^{-1}H)^{-1}F^T . \end{aligned} \tag{58}$$

If

$$M\Lambda M^T = Q + F(H^TR^{-1}H)^{-1}F^T$$

Table 1 Kalman Filter Equations for Recursive Estimation of the Propagation Coefficient Vector, $\alpha_j(k)$

Dynamic System Equation

$$\alpha_j(k+1) = \Phi_j(k+1,k)\alpha_j(k) + \Gamma w_j(k)$$

Measurement Equation

$$z_j(k) = H_j\alpha_j(k) + v_j(k)$$

(j = 1 for Cartesian and j = 2 for slope measurements)

$\hat{\alpha}_j(k/j)$ = Estimate of $\alpha_j(k)$, Given $z_j(j)$

Initial conditions given

$$\hat{\alpha}_j(0/0), \; P_j(0/0)$$

Prior knowledge available

$$[illegible]$$

State extrapolation

$$\hat{\alpha}_j(k+1/k) = \Phi_j(k+1,k)\hat{\alpha}_j(k/k)$$

Covariance extrapolation

$$P_j(k+1/k) = \Phi_j(k+1,k)P_j(k/k)\Phi_j^T(k+1/k) + Q_j$$

Kalman (correction) gain matrix

$$K_j(k+1) = P_j(k+1/k)H_j^T\left[H_jP_j(k+1/k)H_j^T + R_j\right]^{-1}$$

Error covariance update

$$P_j(k+1/k+1) = \left[I_4 - K_j(k+1)H_j\right]P_j(k+1/k)$$

State estimate update

$$\hat{\alpha}_j(k+1/k+1) = \hat{\alpha}_j(k+1/k) + K_j(k)\left[z_j(k+1) - H_j\hat{\alpha}(k+1/k)\right]$$

is the diagonalization of the right hand side of equation [illegible] and [illegible] is the diagonalization of

$$[illegible]$$

then

$$D = [illegible] \qquad (39)$$

[illegible]

[illegible]. CONCLUSIONS

The most significant contributions of this report are the following:

1. A general structure for polynomial curve fitting to noise corrupted measurement data has been developed for both the constant and time varying polynomial coefficient cases;

2. The polynomial fit estimators have been shown to reach the minimum variance bounds for either the maximum likelihood fit estimator when no a priori knowledge is available or the maximum a posteriori fit estimator when prior statistical knowledge on the polynomial coefficients is available;

3. The effect of measurement bias has been analyzed and simple expressions have been obtained for the sensitivity to this bias; and

4. Estimator structures have been developed for two types of measurement data, namely, noisy samples of either the polynomial directly or the polynomial derivative.

The estimation techniques described here should be of considerable assistance in the understanding of physical applications requiring interpolation of data sampled from well-understood dynamical processes.

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